

INDEPENDENCE COMPLEXES OF WELL-COVERED CIRCULANT GRAPHS

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ABSTRACT. We study the independence complexes of families of well-covered circulant graphs discovered by Boros-Gurvich-Milanič, Brown-Hoshino, and Moussi. Because these graphs are well-covered, their independence complexes are pure simplicial complexes. We determine when these pure complexes have extra combinatorial (e.g. vertex decomposable, shellable) or topological (e.g. Cohen-Macaulay, Buchsbaum) structure. We also provide a table of all well-covered circulant graphs on 16 or less vertices, and for each such graph, determine if it is vertex decomposable, shellable, Cohen-Macaulay, and/or Buchsbaum. A highlight of this search is an example of a graph whose independence complex is shellable but not vertex decomposable.

1. INTRODUCTION

Let G be a finite simple graph with vertex set V and edge set E . We say that a subset $W \subseteq V$ is a *vertex cover* of G if $e \cap W \neq \emptyset$ for all edges $e \in E$. The complement of a vertex cover is an *independent set*. A graph G is called *well-covered* if every minimal vertex cover (with respect to the partial order of inclusion) has the same cardinality. Via the duality between vertex covers and independent sets, being well-covered is equivalent to the property that every maximal independent set has the same cardinality. Plummer's survey [24] provides a nice entry point to learn more about well-covered graphs.

Recently, there has been interest in identifying circulant graphs that are well-covered (see, e.g. [2, 3, 4, 17, 22] and some motivation there-within). Recall that a *circulant graph* is defined as follows. Let $n \geq 1$ be an integer, and let $S \subseteq \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. The circulant graph $C_n(S)$ is the graph on the vertex set $V = \{0, 1, \dots, n-1\}$, such that $\{a, b\}$ is an edge of $C_n(S)$ if and only if $|a-b| \in S$ or $n-|a-b| \in S$. Circulant graphs include the family of cycles ($C_n = C_n(\{1\})$) and the family of complete graphs ($K_n = C_n(\{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\})$). For ease of notation, we will write $C_n(a_1, \dots, a_t)$ instead of $C_n(\{a_1, \dots, a_t\})$. The circulant graph $C_{13}(1, 3, 5)$ can be found in Figure 1.

If we consider the *independence complex* of a graph G , i.e., the simplicial complex

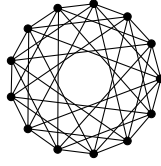
$$\text{Ind}(G) = \{W \subseteq V \mid W \text{ is an independent set of } G\},$$

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FIGURE 1. The circulant graph $C_{13}(1, 3, 5)$.

then the well-coveredness of G is equivalent to the property that $\text{Ind}(G)$ is a pure simplicial complex, that is, all of its maximal faces have the same size. A pure simplicial complex can have additional combinatorial or topological structure; in fact, as summarized in Theorem 2.3, we have the following hierarchy (definitions will be postponed to the next section) of pure simplicial complexes:

$$\text{vertex decomposable} \Rightarrow \text{shellable} \Rightarrow \text{Cohen-Macaulay} \Rightarrow \text{Buchsbaum}.$$

Thus, given a well-covered circulant graph $C_n(S)$, we can ask what additional structure $\text{Ind}(C_n(S))$ entertains. This question is the main focus of this paper. In particular, we look at specific families of well-covered circulant graphs found in [2, 4, 22] and examine the structure of the corresponding independence complex. This paper can be seen as a direct sequel to [29] which considered the same question for some families of well-covered circulants found in [4]. In addition, our work complements the investigation of the topology of independence complex of circulants (e.g., see [1, 18]).

We now provide an overview of the paper. Section 2 provides the relevant background material. In Section 3 we consider the well-covered circulants of the form $C_n(d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor)$ which were characterized by Brown and Hoshino [4]. Theorem 3.3 shows that the independence complexes of these graphs are always Buchsbaum. In addition, we classify when these complexes are vertex decomposable, solving an open problem of [29]. We also include a discussion on the h -vectors of Buchsbaum complexes. In Section 4 we consider circulants of the form $C_n(S)$ where $|S| = \lfloor \frac{n}{2} \rfloor - 1$. The well-covered circulants of this form were characterized by Moussi [22]. Theorem 4.2 refines this result by describing the additional structure of $\text{Ind}(C_n(S))$. In Section 5, we examine one-paired circulants, a family of graphs introduced by Boros, Gurvich, and Milanič [2] as an example of CIS (Cliques Intersecting Stable sets) circulants. We give a new structural result for one-paired circulants (Theorem 5.6), and use this result to determine some properties of its associated independence complex, and to provide a new proof that these circulant graphs are CIS.

In the final two sections we collect together a number of observations and questions based upon a computer search. In particular, we include a table (Table 1) of all well-covered circulants $C_n(S)$ with $n \leq 16$, and for each circulant, determine whether it is vertex decomposable, shellable, etc. As an interesting by-product of this search, we have found that $C_{16}(1, 4, 8)$ is the smallest example of a circulant graph that has a shellable independence complex but is not vertex decomposable. To the best of our knowledge, $C_{16}(1, 4, 8)$ is the first example of a graph with this property (see Remark 6.2).

Although we will not employ this view point, there is an algebraic interpretation of our work. Associated to a graph G on n vertices $V = \{x_1, \dots, x_n\}$ is a quadratic square-free monomial ideal $I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E \rangle$ in $R = k[x_1, \dots, x_n]$. The ideal $I(G)$, commonly called the *edge ideal* of G , is the ideal associated to $\text{Ind}(G)$ via the Stanley-Reisner correspondence. The property that G is well-covered is equivalent to the property that $I(G)$ is an unmixed ideal. Moreover, if $\text{Ind}(G)$ is Cohen-Macaulay or Buchsbaum, the ring $R/I(G)$ also has this property. By identifying Cohen-Macaulay and/or Buchsbaum independence complexes, we are contributing to an ongoing programme in combinatorial commutative algebra to identify graphs G such that $R/I(G)$ is Cohen-Macaulay or Buchsbaum (e.g., see [6, 12, 13, 14, 16, 19, 20, 31]).

Finally, we would like to add a small erratum to [29]. On page 1902, the f -vector and h -vector on line 4 should be $(1, 11, 33, 22)$, respectively, $(1, 8, 14, -1)$; the conclusions are still the same. In Example 6.3, it should read $G[H] = C_{10}(1, 2, 3, 5)$ and $H[G] = C_{10}(1, 4, 5)$, that is, $G[H]$ and $H[G]$ should be reversed. In addition, unknown to us at the time, Theorem 4.65 of Hoshino's Ph.D. thesis [17] contained a proof for the equivalence of Theorem 3.4 (i) and (iv) of [29]. Moreover, we note that Theorem 3.7 of [29] answered Conjecture 4.68 of [17].

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2. BACKGROUND DEFINITIONS AND RESULTS

In this section we review the relevant definitions and results.

A *simplicial complex* on a vertex set $V = \{x_1, \dots, x_n\}$ is a set Δ whose elements are subsets of V such that (a) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$, and (b) for each $i = 1, \dots, n$, $\{x_i\} \in \Delta$. Note that the set $\emptyset \in \Delta$. The independence complex $\text{Ind}(G)$, as defined in the introduction, is a simplicial complex.

An element $F \in \Delta$ is called a *face*. The maximal elements of Δ , with respect to inclusion, are called the *facets* of Δ . If $\{F_1, \dots, F_t\}$ is a complete list of the facets of Δ , we will sometimes write $\Delta = \langle F_1, \dots, F_t \rangle$. The *dimension* of a face $F \in \Delta$, denoted $\dim F$, is given by $\dim F = |F| - 1$, where we make the convention that $\dim \emptyset = -1$. The *dimension of Δ* , denoted $\dim \Delta$, is defined to be $\dim \Delta = \max_{F \in \Delta} \{\dim F\}$. A simplicial complex is *pure* if all of its facets have the same dimension. Note that if $\alpha(G)$ denotes the cardinality of the largest independent set, then $\dim \text{Ind}(G) = \alpha(G) - 1$.

The f -vector of Δ records the number of faces of dimension i of Δ . Precisely, if $\dim \Delta = d$, then the f -vector of Δ is a $(d+2)$ -tuple $f(\Delta) = (f_{-1}, f_0, f_1, \dots, f_d)$ where f_i is the number of faces of dimension i . It follows that $f_{-1} = 1$ and $f_0 = n$. The h -vector of a d -dimensional simplicial complex Δ is a $(d+2)$ -tuple $h(\Delta) = (h_0, h_1, \dots, h_{d+1})$ where

$$(2.1) \quad h_i = \sum_{j=0}^i (-1)^{i-j} \binom{d+1-j}{i-j} f_{j-1}.$$

Given any face $F \in \Delta$, we define the *link* of F in Δ to be

$$\text{link}_\Delta(F) = \{G \in \Delta \mid F \cap G = \emptyset \text{ and } F \cup G \in \Delta\}.$$

The *deletion* of a face F in Δ is the set

$$\text{del}_\Delta(F) = \{G \in \Delta \mid F \not\subseteq G\}.$$

Both $\text{link}_\Delta(F)$ and $\text{del}_\Delta(F)$ are simplicial complexes. When $F = \{x_i\}$, then we simply write $\text{link}_\Delta(x_i)$ or $\text{del}_\Delta(x_i)$.

As promised in the introduction, we now define the relevant pure simplicial complexes.

Definition 2.1. Let $\Delta = \langle F_1, \dots, F_t \rangle$ be a pure simplicial complex on $V = \{x_1, \dots, x_n\}$.

- (i) (see [25]) Δ is *vertex decomposable* if (a) Δ is a simplex, i.e., $\{x_1, \dots, x_n\}$ is the unique maximal facet, or (b) there exists a vertex x such that $\text{link}_\Delta(x)$ and $\text{del}_\Delta(x)$ are vertex decomposable.
- (ii) Δ is *shellable* if there exists an ordering $F_1 < F_2 < \dots < F_t$ such that for all $1 \leq j < i \leq t$, there is some $x \in F_i \setminus F_j$ and some $k \in \{1, \dots, i-1\}$ such that $\{x\} = F_i \setminus F_k$.
- (iii) Δ is *Cohen-Macaulay* if for all $F \in \Delta$, $\tilde{H}_i(\text{link}_\Delta(F), k) = 0$ for all $i < \dim \text{link}_\Delta(F)$. (Here $\tilde{H}_i(-, k)$ denotes the i -th reduced simplicial homology group.)
- (iv) Δ is *Buchsbaum* if $\text{link}_\Delta(x)$ is Cohen-Macaulay for all $x \in V$.

Remark 2.2. Note that for the definition of Cohen-Macaulay, we use the reduced simplicial homology definition due to Reisner, sometimes known as Reisner's Criterion. For an introduction to reduced simplicial homology, see [32, Section 5.2] and for Reisner's Criterion, see [32, Theorem 5.3.5]. If I_Δ is the Stanley-Reisner ideal (see, e.g. [32]) associated to Δ , then one can show that the definitions of Cohen-Macaulay and Buchsbaum presented here are equivalent to the algebraic requirement that R/I_Δ be either Cohen-Macaulay or Buchsbaum, where $R = k[x_1, \dots, x_n]$.

We summarize a number of well-known results about the above simplicial complexes.

Theorem 2.3. Let Δ be a pure simplicial complex on $V = \{x_1, \dots, x_n\}$.

- (i) The following implications hold:
 $\text{vertex decomposable} \Rightarrow \text{shellable} \Rightarrow \text{Cohen-Macaulay} \Rightarrow \text{Buchsbaum}.$
- (ii) If $\dim \Delta = 0$, then Δ is vertex decomposable (and thus, shellable, Cohen-Macaulay, and Buchsbaum).
- (iii) If $\dim \Delta = 1$, then Δ is vertex decomposable/shellable/Cohen-Macaulay if and only if Δ is connected. If Δ is not connected, then Δ is Buchsbaum but not Cohen-Macaulay.
- (iv) If $\dim \Delta \geq 2$ and Δ is Cohen-Macaulay, then Δ is connected.
- (v) If Δ is Cohen-Macaulay, then $h(\Delta)$ has no negative entries.

Proof. (i) Vertex decomposability implies shellability by [25, Corollary 2.9]; shellability implies Cohen-Macaulay by [32, Theorem 5.3.18], and Cohen-Macaulay implies Buchsbaum since $\text{link}_\Delta(F)$ is Cohen-Macaulay for all faces $F \in \Delta$ when Δ is Cohen-Macaulay by [32, Proposition 5.3.8]. (ii) is [25, Proposition 3.1.1], and (iii) is [25, Theorem 3.1.2]. Because $\text{link}_\Delta(\emptyset) = \Delta$, $\tilde{H}_0(\Delta, k) = 0$ since $\dim \Delta \geq 2$ and Δ is Cohen-Macaulay. By [32, Proposition 5.2.3], $\tilde{H}_0(\Delta, k) + 1$ is the number of connected components of Δ , i.e., Δ is connected. For (v) see [32, Theorem 5.4.8]. \square

Because we are interested in $\text{Ind}(G)$ for graphs G , we will say G is vertex decomposable, shellable, Cohen-Macaulay, or Buchsbaum if $\text{Ind}(G)$ has the corresponding property. Except for the Buchsbaum property, it is enough to consider connected components of G .

Lemma 2.4. *Suppose that G and H are two disjoint graphs that are both vertex decomposable/shellable/Cohen-Macaulay. Then $G \cup H$ is vertex decomposable/shellable/Cohen-Macaulay.*

Proof. See [34, Lemma 20] which states a more general result (i.e., for non-pure vertex decomposability and non-pure shellability). The Cohen-Macaulay case can also be found in [32, Proposition 6.2.8]. \square

The union of two or more Buchsbaum graphs will fail to be Buchsbaum.

Lemma 2.5. *Let G and H be two disjoint graphs that are both Buchsbaum, but not Cohen-Macaulay. Then $G \cup H$ is not Buchsbaum.*

Proof. Let $\text{Ind}(G)$ and $\text{Ind}(H)$ be the independence complexes associated to G and H . The join of these two simplicial complexes gives us the independence complex of $G \cup H$:

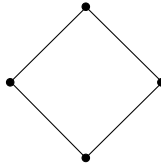
$$\text{Ind}(G \cup H) = \text{Ind}(G) \star \text{Ind}(H) = \{F \cup E \mid F \in \text{Ind}(G) \text{ and } E \in \text{Ind}(H)\}.$$

Let x be any vertex of $G \cup H$. Without loss of generality, assume that x is in G . Then

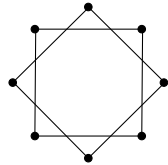
$$\text{link}_{\text{Ind}(G \cup H)}(x) = \text{link}_{\text{Ind}(G)}(x) \star \text{Ind}(H).$$

By [32, Proposition 5.3.16], the join of two simplicial complexes is Cohen-Macaulay if and only if both complexes are Cohen-Macaulay. But $\text{Ind}(H)$ is not Cohen-Macaulay, so $\text{link}_{\text{Ind}(G \cup H)}(x)$ cannot be Cohen-Macaulay, and thus, $G \cup H$ cannot be Buchsbaum. \square

Example 2.6. Consider the four-cycle $G = C_4$, that is, the circulant graph $C_4(1)$:



If we label the vertices 0, 1, 2, 3 in clockwise order, then $\text{Ind}(G) = \langle \{0, 2\}, \{1, 3\} \rangle$ is disconnected. Since $\dim \text{Ind}(G) = 1$, by Theorem 2.3 (iii) that $C_4(1)$ is Buchsbaum, but not Cohen-Macaulay. In fact, $C_4(1)$ is the smallest well-covered circulant graph with this property. The graph $C_8(2)$ consists of two disjoint copies of $C_4(1)$:



By Lemma 2.5, $C_8(2)$ cannot be Buchsbaum. In fact, $C_8(2)$ is the smallest well-covered circulant that is not Buchsbaum. An example of a connected well-covered circulant that is not Buchsbaum is presented in Theorem 6.1

The following lemma will simplify some of our future arguments.

Lemma 2.7. *Let $G = C_n(S)$ be a well-covered circulant graph.*

- (i) *If $\text{link}_{\text{Ind}(G)}(0)$ is Cohen-Macaulay, then G is Buchsbaum.*
- (ii) *If $\dim \text{Ind}(G) = 1$, then G is Buchsbaum.*
- (iii) *If $\dim \text{Ind}(G) = 2$, then G is Buchsbaum if $\text{link}_{\text{Ind}(G)}(0)$ is connected.*

Proof. (i) By the symmetry of the graph, $\text{link}_{\text{Ind}(G)}(0) \cong \text{link}_{\text{Ind}(G)}(i)$ for all vertices $i \in \{1, \dots, n-1\}$. So, to check if G is Buchsbaum, it suffices to check that $\text{link}_{\text{Ind}(G)}(0)$ is Cohen-Macaulay. For (ii), respectively (iii), we use the fact that $\dim \text{link}_{\text{Ind}(G)}(0) = 0$, respectively, 1, and then apply Theorem 2.3 (ii), respectively, (iii). \square

3. CIRCULANTS OF THE FORM $C_n(d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor)$

In this section we determine the properties of the independence complex of well-covered circulants of the form $C_n(d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor)$ with $d \geq 1$. These graphs are sometimes called the complement of the powers of cycles because they are the complement of $C_n(1, 2, \dots, d)$.

Brown and Hoshino determined all the values of n and d such that $G = C_n(d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor)$ is well-covered, i.e., $\text{Ind}(G)$ is a pure simplicial complex:

Theorem 3.1 ([4, Theorem 4.2]). *Let n and d be integers with $n \geq 2d+2$ and $d \geq 1$. Then $C_n(d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor)$ is well-covered if and only if $n > 3d$ or $n = 2d+2$.*

The f -vectors and h -vectors for $\text{Ind}(C_n(d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor))$ for some n and d are given below.

Lemma 3.2. *Let n and d be integers with $n > 3d$ and $d \geq 1$. If $G = C_n(d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor)$, then the f -vector of $\text{Ind}(G)$ is given by*

$$f(\text{Ind}(G)) = \left(1, n, \binom{d}{1}n, \binom{d}{2}n, \binom{d}{3}n, \dots, \binom{d}{d-1}n, \binom{d}{d}n \right).$$

Consequently, $h(\text{Ind}(G))$, the h -vector of $\text{Ind}(G)$, is

$$\left(1, n - (d+1), (-1)^2 \binom{d+1}{2}, (-1)^3 \binom{d+1}{3}, \dots, (-1)^d \binom{d+1}{d}, (-1)^{d+1} \binom{d+1}{d+1} \right).$$

Proof. By [4, Theorem 3.2], the independence polynomial of $G = C_n(d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor)$ when $n > 3d$ and $d \geq 1$ is given by

$$I = I\left(C_n\left(d+1, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right), x\right) = 1 + nx(1+x)^d.$$

The coefficient of x^i in $I(G, x)$ counts the number of independent sets of size i in G . So, the coefficient of x^i is precisely f_{i-1} , the number of faces of $\text{Ind}(G)$ of dimension $(i-1)$. Thus $f(\text{Ind}(G))$ can now be computed by expanding out the polynomial I .

The h -vector of $\text{Ind}(G)$ is computed from the f -vector using (2.1). We omit the details, but we note that the details can be found in [17, Theorem 4.64]. \square

The main result of this section refines Theorem 3.1; in particular, all the pure independence complexes of Theorem 3.1 are either vertex decomposable or Buchsbaum.

Theorem 3.3. *Let n and d be integers with $n \geq 2d + 2$ and $d \geq 1$. The following are equivalent*

- (i) $C_n(d + 1, d + 2, \dots, \lfloor \frac{n}{2} \rfloor)$ is Buchsbaum.
- (ii) $C_n(d + 1, d + 2, \dots, \lfloor \frac{n}{2} \rfloor)$ is well-covered.
- (iii) $n > 3d$ or $n = 2d + 2$.

Furthermore, $C_n(d + 1, d + 2, \dots, \lfloor \frac{n}{2} \rfloor)$ is vertex decomposable/shellable/Cohen-Macaulay if and only if $n = 2d + 2$ and $d \geq 1$, or $d = 1$ and $n > 3$.

Proof. Let $G = C_n(d + 1, d + 2, \dots, \lfloor \frac{n}{2} \rfloor)$ with $n \geq 2d + 2$ and $d \geq 1$.

The equivalence of (ii) and (iii) is simply Theorem 3.1. Furthermore, if G is Buchsbaum, then G must be well-covered, so (i) implies (ii). It suffices to show that if G is well-covered, it is also Buchsbaum.

If $n = 2d + 2$, then $G = C_{2d+2}(d + 1)$, which implies that G is $(d + 1)$ disjoint copies of the complete graph K_2 . Since a K_2 is vertex decomposable, G must be vertex decomposable (Lemma 2.4), and hence Buchsbaum.

So, suppose that $n > 3d$ and $d \geq 1$. We first note that $\text{Ind}(G)$ has dimension d from its f -vector in Lemma 3.2. Each element of the set

$$\{\{i, i + 1, i + 2, \dots, i + d\} \mid 0 \leq i \leq n - 1\}$$

where the indices are computed modulo n , is an independent set in G . Because $f_d = n$, and because each element of the above set is distinct, these elements form a complete list of the facets of $\text{Ind}(G)$.

If $d = 1$, then $\text{Ind}(G)$ is connected, so it is vertex decomposable by Theorem 2.3 and hence Buchsbaum. If $d > 1$, the facets

$$\{n - d, n - d + 1, \dots, 0\}, \{n - d + 1, n - d + 2, \dots, 0, 1\}, \dots, \{0, 1, \dots, d\}$$

are a complete list of the facets that contain 0. Thus, the facets of $\text{link}_{\text{Ind}(G)}(0)$ are

$$\{n - d, n - d + 1, \dots, n - 1\}, \{n - d + 1, n - d + 2, \dots, n - 1, 1\},$$

$$\{n - d + 2, n - d + 3, \dots, n - 1, 1, 2\}, \dots, \{1, \dots, d\}.$$

It follows from the order in which we have written these facets that $\text{link}_{\text{Ind}(G)}(0)$ is shellable, and hence, by Theorem 2.3 Cohen-Macaulay. So, G is Buchsbaum by Lemma 2.7.

Observe that when $n = 2d + 2$, or $n > 3$ and $d = 1$, then $\text{Ind}(G)$ is vertex decomposable, and so vertex decomposable, shellable, and Cohen-Macaulay. On the other hand, if $n > 3d$ and $d \geq 2$, then by Lemma 3.2, the entry h_3 of $h(\text{Ind}(G))$ is negative. Thus, by Theorem 2.3 (v), G is not Cohen-Macaulay (and thus, not vertex decomposable or shellable either). This completes the proof of the final statement. \square

Remark 3.4. Hoshino [17, Theorem 4.64] first characterized when $C_n(d + 1, \dots, \lfloor \frac{n}{2} \rfloor)$ is shellable. Our theorem shows that the independence complex still has some structure when the graph is not shellable, and moreover, it has a stronger structure if it is shellable.

Remark 3.5. Hibi [15] is attributed with first asking for a characterization of the h -vectors of Buchsbaum simplicial complexes. This question remains open (see [23, 28] for some work on this problem). However, by the above theorem and Lemma 3.2,

$$\left(1, n - (d + 1), \binom{d + 1}{2}, -\binom{d + 1}{3}, \dots, (-1)^d \binom{d + 1}{d}, (-1)^{d+1} \binom{d + 1}{d + 1}\right)$$

is a valid h -vector of a d -dimensional Buchsbaum simplicial complex on n vertices with $n > 3d$ and $d \geq 2$.

4. CIRCULANTS OF THE FORM $C_n(1, \dots, \hat{i}, \dots, \lfloor \frac{n}{2} \rfloor)$

Moussi's thesis [22] contains a number of families of well-covered circulants. We analyze the family $G = C_n(S)$ with $|S| = \lfloor \frac{n}{2} \rfloor - 1$. As shown in [22], all circulants in this family are well-covered (below, $\alpha(G)$ denotes the size of the largest independent set of G):

Theorem 4.1 ([22, Theorem 6.4]). *Let $G = C_n(S)$ be the circulant graph with $S = \{1, \dots, \hat{i}, \dots, \lfloor \frac{n}{2} \rfloor\}$ for any $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Then G is well-covered. Furthermore, $\alpha(G) = 2$ except if $i = \frac{n}{3}$, in which case, $\alpha(G) = 3$.*

As in the previous section, the well-covered circulants in this family can be divided into two groups, those that are vertex decomposable and those that are merely Buchsbaum.

Theorem 4.2. *Let $G = C_n(S)$ be the circulant graph with $S = \{1, \dots, \hat{i}, \dots, \lfloor \frac{n}{2} \rfloor\}$ for any $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Then G is Buchsbaum. Furthermore G is vertex decomposable/shellable/Cohen-Macaulay if and only if $\gcd(i, n) = 1$.*

Proof. Let $G = C_n(1, \dots, \hat{i}, \dots, \lfloor \frac{n}{2} \rfloor)$ for some $i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. By Theorem 4.1, $\text{Ind}(G)$ is pure.

If $i \neq \frac{n}{3}$, then $\dim \text{Ind}(G) = 1$ since $\alpha(G) = 2$ by Theorem 4.1. Thus, G is Buchsbaum by Lemma 2.7 (ii). If $i = \frac{n}{3}$, the $\dim \text{Ind}(G) = 2$. In particular,

$$(4.1) \quad \text{Ind}(G) = \langle \{0, i, 2i\}, \{1, i + 1, 2i + 1\}, \dots, \{i - 1, 2i - 1, 3i - 1\} \rangle.$$

Because $\text{link}_{\text{Ind}(G)}(0) = \langle \{i, 2i\} \rangle$ is connected, we can apply Lemma 2.7. So, G is always Buchsbaum.

We now prove the second statement. We treat the cases $i \neq \frac{n}{3}$ and $i = \frac{n}{3}$ separately.

If $i \neq \frac{n}{3}$, then $\dim \text{Ind}(G) = 1$. So, by Theorem 2.3, it suffices to show that $\text{Ind}(G)$ is connected if and only if $\gcd(i, n) = 1$.

If $\gcd(i, n) = 1$, then the map $\phi : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ given by $\phi(j) = ji$ is a bijection. So, the elements $\{0, i, 2i, 3i, \dots, (n - 1)i\}$ are all distinct elements. But this implies that

$$\{0, i\}, \{i, 2i\}, \{2i, 3i\}, \dots, \{(n - 1)i, 0\}$$

is path of facets in $\text{Ind}(G)$ that includes all the vertices, and hence $\text{Ind}(G)$ is connected.

On the other hand, suppose that $\gcd(i, n) = k > 1$. Then $\{0, i, 2i, \dots, (n/k - 1)i\}$ and $\{1, i + 1, \dots, (n/k - 1)i + 1\}$ are disjoint sets in \mathbb{Z}_n . Now every vertex a is non-adjacent

in G to exactly two vertices, namely $a + i$ and $a - i$ (modulo n). But then there is no path from the connected facets

$$\{0, i\}, \{i, 2i\}, \dots, \{(n/k - 1)i, 0\}$$

to any of the connected facets

$$\{1, i + 1\}, \{i + 1, 2i + 1\}, \dots, \{(n/k - 1)i + 1, 1\}.$$

In other words, $\text{Ind}(G)$ is disconnected.

If $i = \frac{n}{3}$, then $\gcd(i, n) = 1$ if and only if $i = 1$, i.e., $n = 3$. The facets of $\text{Ind}(G)$ are given in (4.1). If $i = 1$ and $n = 3$, then $\text{Ind}(G)$ is simply the simplex with unique maximal facet $\{0, 1, 2\}$, and thus, it is vertex decomposable. If $i > 1$, then $\text{Ind}(G)$ is a disconnected simplicial complex of dimension two, so by Theorem 2.3 (iv), it is not Cohen-Macaulay, and thus, not vertex-decomposable. \square

5. ONE-PAIRED CIRCULANTS

In [2], Boros *et al.* studied circulant graphs for which every maximal clique (a clique is a subgraph in which every vertex is adjacent to every other vertex) intersects each maximal independent set. Such a graph is called a CIS graph:

Definition 5.1. A graph G is *CIS* if for every maximal clique C and every maximal independent set I in G , $C \cap I \neq \emptyset$. (CIS is an acronym for Cliques Intersect Stable sets).

Boros *et al.* [2, Theorem 3] showed that CIS circulants are well-covered. In fact, the main theorem of [2] is a classification of circulant graphs that are CIS; precisely, the circulant graph G is CIS if and only if all maximal independent sets have size $\alpha(G)$ and all maximal cliques have size $\omega(G)$, and $\alpha(G)\omega(G) = |V(G)|$.

In [2], the authors also describe how to construct some CIS circulants graphs. One construction is the one-paired circulants described below. We provide a more direct proof that one-paired circulants are CIS (Corollary 5.7) by first characterizing their structure (Theorem 5.4). We then consider the independence complex of a one-paired circulant.

Definition 5.2. The circulant graph $G = C_n(S)$ is *one-paired* if there exists an ordered pair of positive integers (a, b) such that $ab|n$ and $S = \{d \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\} : a|d \text{ and } ab \nmid d\}$. We denote the one-paired circulant by $G = C(n; a, b)$.

Example 5.3. Let $n = 12$ and consider the ordered pair $(2, 3)$. Then $C(12; 2, 3)$ is the circulant graph $C_{12}(2, 4)$. Compare this graph to $C(12; 3, 2)$ which is $C_{12}(3)$.

We begin with a structural result for $C(n; a, b)$. Note that $G \vee H$, the *join* of G and H , is the graph with vertex set $V_G \cup V_H$ and edges $E_G \cup E_H \cup \{\{x, y\} \mid x \in V_G \text{ and } y \in V_H\}$.

Theorem 5.4. Let $G = C(n; a, b)$ be a one-paired circulant. Then

$$C(n; a, b) = \bigcup_{i=1}^a \left(\bigvee_{j=1}^b \overline{K_{\frac{n}{ab}}} \right) \text{ and } \alpha(G) = \frac{n}{b}.$$

Proof. Let $G = C(n; a, b)$ be a one-paired circulant with $\alpha = \alpha(G)$. By the definition of a one-paired circulant, $n = kab$ for some positive integer k . Let $I = \{0, ab, 2ab, \dots, (k-1)ab\}$. Note that I is an independent set of size k in G . By vertex transitivity of G , the cosets in $W = \{I, I + a, I + 2a, \dots, I + (b-1)a\}$ are b disjoint independent sets of size k . We claim that the subgraph of G induced by W is $\bigvee_{j=1}^b \overline{K_k}$. In particular, suppose $m_i \in I + ia$ and $m_r \in I + ra$ for some $i, r \in \{0, 1, \dots, (b-1)\}$, $i > r$. Then $a \mid (m_i - m_r)$ but $ab \nmid (m_i - m_r)$ since $(i - r) < b$. It follows that m_i is adjacent to m_r in G . Hence the claim is established.

The cosets $W, W + 1, \dots, W + (a-1)$ form a partition of the vertex set of G into disjoint graphs on kb vertices. Thus $G = \bigcup_{i=1}^a \left(\bigvee_{j=1}^b \overline{K_k} \right)$. It follows that $\alpha(G) = ak = n/b$. \square

Setting $a = 1$ in Theorem 5.4 gives us the following corollary:

Corollary 5.5.

$$C(n; 1, b) = \bigvee_{j=1}^b \overline{K_{\frac{n}{b}}}.$$

Re-combining with Theorem 5.4, we can re-characterize the structure of one-paired circulants:

Corollary 5.6.

$$C(n; a, b) = \bigcup_{i=1}^a C\left(\frac{n}{a}; 1, b\right).$$

As a result of Corollary 5.6, when exploring one-paired circulants, we can focus on the the graphs $C(n; 1, b)$. We can determine the clique number of a one-paired circulant. In fact, we have a more direct proof of [2, Theorem 4] that every one-paired circulant is CIS.

Corollary 5.7. *Let $G = C(n; a, b)$ be a one-paired circulant with $\alpha = \alpha(G)$. Then $\omega(G) = b$, $n = \alpha b$, and G is a CIS graph. Furthermore, G is well-covered.*

Proof. By Corollary 5.6, a maximum clique in G must be a maximum clique in $C(\frac{n}{a}; 1, b)$. By Corollary 5.5, it follows that $\omega(G) = b$. Thus by Theorem 5.4, $n = \alpha \omega$. It also follows from the structure of G that each maximal independent set intersects each clique of size ω , so G is a CIS graph. In addition, these independent sets all have the same size, i.e., G is well-covered. \square

We can also determine the f -vector of the independence complex $\text{Ind}(C(n; 1, b))$ (equivalently, the independence polynomial of $C(n; 1, b)$) directly from the structural description.

Theorem 5.8. *Let $G = C(n; 1, b) = C(mb, 1, b)$. Then the f -vector of $\text{Ind}(G)$ is*

$$f(\text{Ind}(G)) = \left(1, \binom{m}{1}b, \binom{m}{2}b, \dots, \binom{m}{m-1}b, b \right).$$

Proof. This will be a counting argument based upon our description of the graph $C(n; 1, b)$. Suppose $G = C(mb, 1, b)$. By Corollary 5.5, $G = \bigcup_{j=1}^b \overline{K_m}$. (Note that $\bigcup_{j=1}^b \overline{K_m} =$

$\bigvee_{j=1}^b \overline{K_m}$.) Hence, any non-adjacency is found within a $\overline{K_m}$. Thus, for each independent set of size k , there will be $\binom{m}{k}$ choices of vertices in each of the b copies of $\overline{K_m}$. Hence, there are $\binom{m}{k}b$ independence sets of size k . Thus, $f(\text{Ind}(G)) = (1, \binom{m}{1}b, \binom{m}{2}b, \dots, \binom{m}{m-1}b, b)$. \square

Theorem 5.9. *Let G be the one-paired circulant $G = C(mb; 1, b)$. Then G is Buchsbaum. Furthermore, G is vertex decomposable/shellable/Cohen-Macaulay if and only if $m = 1$.*

Proof. Suppose $G = C(mb, 1, b)$. By Theorem 5.8, $\dim \text{Ind}(G) = m - 1$ with b facets. So,

$$\{\{0 + j, b + j, 2b + j, \dots, (m - 1)b + j\} \mid j = 0, \dots, b - 1\}$$

is a complete list of the facets of $\text{Ind}(G)$, where addition is modulo n .

Since 0 only appears in the facet $\{0, b, 2b, \dots, (m - 1)b\}$, $\text{link}_{\text{Ind}(G)}(0)$ is the simplex $\langle \{b, 2b, \dots, (m - 1)b\} \rangle$, and thus the link is vertex decomposable. So G is Buchsbaum by Lemma 2.7.

If $m > 1$, then the facets of $\text{Ind}(G)$ are disjoint, and thus $\text{Ind}(G)$ is not connected. As a consequence, G is not vertex decomposable, shellable, or Cohen-Macaulay by Theorem 2.3 (iii) and (iv). However, If $m = 1$, then $\text{Ind}(G)$ has dimension 0, and so is vertex decomposable, shellable, and Cohen-Macaulay by Theorem 2.3 (ii). \square

Corollary 5.10. *Let G be the one-paired circulant $G = C(n; a, b)$.*

- (i) *G is vertex decomposable/shellable/Cohen-Macaulay if and only if $n = ab$.*
- (ii) *G is Buchsbaum but not Cohen-Macaulay if and only if $a = 1$ and $ab < n$.*
- (iii) *$\text{Ind}(G)$ is pure but not Buchsbaum if and only if $1 < a$ and $ab < n$.*

Proof. (i) By Corollary 5.6, $C(n; a, b)$ is the disjoint union of a copies of $C(\frac{n}{a}; 1, b)$. By Lemma 2.4 $C(n; a, b)$ will be vertex decomposable, shellable, or Cohen-Macaulay if and only if each connected component $C(\frac{n}{a}; 1, b)$ has this property. But by Theorem 5.9, $C(\frac{n}{a}; 1, b)$ has these properties if and only if $\frac{n}{a} = b$, i.e., $n = ab$.

We now prove (ii) and (iii). Note that $\text{Ind}(G)$ is pure by Corollary 5.7, so it suffices to show that $\text{Ind}(G)$ is Buchsbaum, but not Cohen-Macaulay, if and only if $a = 1$. If $a = 1$, then $G = C(n; a, b) = C(n; 1, b)$. By Theorem 5.9, G is Buchsbaum. In addition, if $b < n$, G cannot be Cohen-Macaulay.

Now suppose that $a \geq 2$ and $ab < n$. By Theorem 5.9 and Corollary 5.6, $C(n; a, b)$ is the disjoint union of $a \geq 2$ Buchsbaum graphs that are not Cohen-Macaulay. Then by Lemma 2.5, $C(n; a, b)$ cannot be Buchsbaum. \square

Example 5.11. The graph $C_8(2)$ equals the one-paired circulant $C(8; 2, 2)$. Since $2 \cdot 2 < 8$ and $1 < 2$, we have that $C_8(2)$ is pure but not Buchsbaum by the above result. This was already noted in Example 2.6.

6. MINIMAL EXAMPLES

Recall that the following implications always hold for pure simplicial complexes:

$$\text{vertex decomposable} \Rightarrow \text{shellable} \Rightarrow \text{Cohen-Macaulay} \Rightarrow \text{Buchsbaum}.$$

For many families of well-covered graphs, the reverse implications also hold, for example, see [7, 14, 34] for chordal graphs and [9, 30] for bipartite graphs. However, if we restrict to well-covered circulants, the reverse implications may fail to hold.

Example 2.6 already gives an example of a well-covered circulant that is not Buchsbaum. As we detail in the next section, we used computer algebra systems to determine all well-covered circulants on $n \leq 16$ vertices (see Table 1). Our computer search found the following vertex minimal counter-examples to the reverse implications.

- Theorem 6.1.** (i) *The disconnected graph $C_8(2)$ is the smallest well-covered circulant that is not Buchsbaum. The well-covered circulant $C_{10}(1, 4)$ is the smallest connected well-covered circulant that is not Buchsbaum.*
- (ii) *The graph $C_4(1)$ is the smallest well-covered circulant that is Buchsbaum but not Cohen-Macaulay.*
- (iii) *The graph $C_{16}(1, 4, 8)$ is the smallest well-covered circulant that is shellable but not vertex decomposable.*

Proof. The minimality is a result of our computer search. Example 2.6 showed that $C_4(1)$ is Buchsbaum but not Cohen-Macaulay and $C_8(2)$ is not Buchsbaum. There are only two facets of $\Delta = \text{Ind}(C_{10}(1, 4))$ that contain 0, namely $\{0, 3, 5, 8\}$ and $\{0, 2, 5, 7\}$. So $\text{link}_\Delta(0) = \langle \{3, 5, 8\}, \{2, 5, 7\} \rangle$. But then the link has f -vector $(1, 5, 6, 2)$ and h -vector $(1, 2, -1, 0)$, so the link is not Cohen-Macaulay, and thus, $C_{10}(1, 4)$ is not Buchsbaum.

For $G = C_{16}(1, 4, 8)$ (see Figure 2), the deletion $\text{del}_{\text{Ind}(G)}(0)$ has f -vector $(1, 15, 70, 117, 60)$,

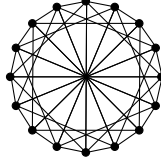


FIGURE 2. The circulant graph $C_{16}(1, 4, 8)$.

and thus, h -vector $(1, 11, 31, 18, -1)$. So, by Theorem 2.3 (v), the deletion is not Cohen-Macaulay, so the deletion cannot be vertex decomposable. Because of the symmetry of the graph, G has no vertex x such that $\text{del}_{\text{Ind}(G)}(x)$ is vertex decomposable, and so G cannot be vertex decomposable. The simplicial complex $\text{Ind}(G)$ has 80 facets. For completeness, here is one shelling order read left to right (the independence complex has dimension three, so each group of four integers represents a facet):

8 10 13 15	4 10 13 15	6 8 13 15	2 8 13 15	4 6 13 15	2 4 13 15	5 8 10 15
5 10 12 15	1 10 12 15	6 9 12 15	1 6 12 15	1 8 10 15	1 4 10 15	4 6 9 15
2 4 9 15	2 9 12 15	2 5 12 15	1 6 8 15	2 5 8 15	1 4 6 15	6 8 11 13
4 9 11 14	7 9 12 14	3 9 12 14	0 9 11 14	4 7 9 14	0 7 9 14	0 3 9 14
2 8 11 13	2 7 9 12	2 4 7 13	4 7 10 13	5 7 10 12	5 7 12 14	3 5 12 14
0 5 7 14	0 5 11 14	5 8 11 14	3 5 8 14	0 3 5 14	1 7 10 12	1 7 12 14
1 3 12 14	1 8 11 14	1 3 8 14	3 5 10 12	3 8 10 13	3 6 8 13	1 3 10 12
2 5 7 12	0 6 9 11	0 6 11 13	0 2 11 13	0 7 10 13	0 2 7 13	4 6 9 11
4 6 11 13	2 4 11 13	2 4 9 11	0 2 9 11	1 6 8 11	2 5 8 11	1 4 6 11
1 4 11 14	1 4 7 14	0 2 5 11	3 5 8 10	1 3 8 10	0 5 7 10	1 4 7 10
0 3 5 10	0 3 10 13	0 3 6 13	2 4 7 9	0 2 7 9	0 3 6 9	3 6 9 12

1 3 6 12 1 3 6 8 0 2 5 7

□

Remark 6.2. To the best of our knowledge, $C_{16}(1, 4, 8)$ is the first example of an independence complex that is shellable but not vertex decomposable. Provan and Billera's original paper on vertex decomposability (see [25]) points out that Walkup's example (see [33]) of a simplicial complex on 56 vertices and over 8000 27-simplicies is an example of a shellable but not vertex decomposable simplicial complex. (Provan and Billera proved that any vertex decomposable simplicial complex satisfies Hirsh's conjecture, while Walkup's example was a counter-example to this example.) The short note of Moriyama and Takeuchi [21] contains a list of the minimal two dimensional simplicial complexes that are vertex decomposable, but not shellable. We checked these complexes, but none of them are the independence complex of a graph. Our example is also interesting for the following reason. It is known from [25] that the barycentric subdivision of any shellable simplicial complex is vertex decomposable. So, the independence complex of $C_{16}(1, 4, 8)$ cannot be constructed by taking the barycentric subdivision of a shellable simplicial complex.¹

Remark 6.3. The *lexicographical product* of the graphs G and H , denoted $G[H]$, is the graph with the vertex set $V_G \times V_H$ and where (w, x) and (y, z) are adjacent if $\{w, y\} \in E_G$ or if $w = y$, then $\{x, z\} \in E_H$. In a forthcoming paper, we will explore how the topological properties (e.g., vertex decomposable, shellable) of $\text{Ind}(G)$ and $\text{Ind}(H)$ are preserved under the lexicographical product. As an application we will use the graph $C_{16}(1, 4, 8)$ to build an infinite family of graphs which are shellable but not vertex decomposable.

The reader will notice that there is no example of a well-covered circulant that is Cohen-Macaulay but not shellable. We know of no such a graph, so we leave it as a question:

Question 6.4. *Is there a well-covered circulant that is Cohen-Macaulay but not shellable?*

In fact, we are not aware of any graph whose independence complex is Cohen-Macaulay but not shellable.

7. ADDITIONAL OBSERVATIONS AND COMPUTATIONS

Using *Macaulay2* [11] and Sage [27] we determined all well-covered circulants on $n \leq 16$ vertices, and determined their combinatorial topological properties. This information is collated in Table 1 at the end of the section. We record a number of observations about the independence complex of well-covered circulants, some of which are based upon our computer experiments.

¹After submitting this paper, we observed that $C_{16}(1, 4, 8)$ also gives a negative answer to [31, Conjecture 2]. R. Villarreal conjectured that every Cohen-Macaulay graph G has a vertex x such that $G \setminus x$ is also Cohen-Macaulay. It was already known that this conjecture is false due to an example of Terai [32, Exercise 6.2.24]. However Terai's example does not hold for all characteristics. Our example works in all characteristics. To see why, note that $C_{16}(1, 4, 8)$ is shellable, so it is Cohen-Macaulay over any field. As we observed in the proof of Theorem 6.1, for any vertex x of $C_{16}(1, 4, 8)$, the h -vector of the independence complex of $C_{16}(1, 4, 8) \setminus x$, which only depends upon the combinatorics of the complex, implies that $C_{16}(1, 4, 8) \setminus x$ is not Cohen-Macaulay.

7.1. 1-well-covered circulants. A well-covered graph G is said to be *1-well-covered* if $G \setminus \{x\}$, the graph with the vertex x and of its adjacent edges removed, is a well-covered graph for all vertices $x \in V$. This notion was introduced by Staples [26]. In [22, Theorem 3.3], Moussi determined which of the well-covered circulants of the form $G = C_n(1, 2, \dots, d)$ were also 1-well-covered.

What is striking about [22, Theorem 3.3] is that the class of 1-well-covered circulants of the form $G = C_n(1, 2, \dots, d)$ coincides exactly with those that are vertex decomposable, as first found in [29, Theorem 3.4]. This observation suggests some connection between the two concepts. Indeed, vertex decomposability implies 1-well-coveredness:

Theorem 7.1. *Let G be a circulant graph. If G is vertex decomposable, then G is 1-well-covered.*

Proof. We begin with the straight-forward observation that for any graph G , we have $\text{del}_{\text{Ind}(G)}(x) = \text{Ind}(G \setminus \{x\})$ for any vertex x of G . Now, if G is vertex decomposable, then there exists a vertex x such that $\text{del}_{\text{Ind}(G)}(x)$ is vertex decomposable, hence pure. But by symmetry, all vertices x will have this property. By the above observation, this means that $\text{Ind}(G \setminus x)$ is pure, i.e., $G \setminus x$ is well-covered for all x . \square

The above result allows us to give a new proof for [22, Theorem 3.3].

Theorem 7.2. *Let n and d be integers with $n \geq 2d \geq 2$ and let $G = C_n(1, 2, \dots, d)$. Then G is 1-well-covered if and only if $n \leq 3d + 2$ and $n \neq 2d + 2$.*

Proof. By [4, Theorem 4.1], G is well-covered if and only if $n \leq 3d + 2$ or $n = 4d + 3$. Since a 1-well-covered graph must also be well-covered, we only need to look at the cases $n \leq 3d + 2$ or $n = 4d + 3$. If $n \leq 3d + 2$ and $n \neq 2d + 2$, then G is vertex decomposable by [29, Theorem 3.4], so by Theorem 7.1, G is 1-well-covered.

It suffices to show if $n = 2d + 2$ or $n = 4d + 3$, G is not 1-well-covered. If $n = 2d + 2$, consider the graph $G \setminus \{0\}$. Then the vertex $d + 1$ is adjacent to every other vertex, so $\{d + 1\}$ is a maximal independent set. However, $\{1, d + 2\}$ is also an independent set, so $G \setminus \{0\}$ is not well-covered. When $n = 4d + 3$, we again consider the graph $G \setminus \{0\}$ which is a graph on the vertices $\{1, 2, \dots, 4d + 2\}$. The set $\{d + 1, 3d + 2\}$ is a maximal independent set in this graph. To see this, note that $d + 1$ is adjacent to $\{1, \dots, d, d + 2, \dots, 2d + 1\}$, and $3d + 2$ is adjacent to $\{2d + 2, \dots, 3d - 1, 3d + 1, \dots, 4d + 2\}$. On the other hand, $\{1, d + 2, 3d + 3\}$ is an independent set of size 3, so $G \setminus \{0\}$ is not well-covered. \square

Example 7.3. When we computed our table of well-covered circulants on $n \leq 16$ vertices, we also checked which of these circulants were 1-well-covered (see Table 1). In particular, by Table 1, the converse of Theorem 7.1 is false. The graph $C_{10}(1, 2, 3, 5)$ is Buchsbaum but not vertex decomposable. However, it is still 1-well-covered. Furthermore, this is the minimal such example with respect to the number of vertices. Note that the fact that $C_{10}(1, 2, 3, 5)$ is Buchsbaum but not vertex decomposable can also be deduced from Theorem 4.2 since $\gcd(4, 10) \neq 1$.

We pose the following question based upon our computations:

Question 7.4. *Can the hypotheses of Theorem 7.1 be relaxed, i.e., does the conclusion still hold if G is shellable or Cohen-Macaulay?*

One of the difficulties in answering this question is that in most cases in which we find a shellable or Cohen-Macaulay circulant graph, it is also vertex decomposable. Our new example of a graph that is shellable but not vertex decomposable, i.e., the graph $C_{16}(1, 4, 8)$, is also 1-well-covered.

7.2. Circulant Cubic Graphs. A *cubic* graph is a graph such that every vertex has degree three. Brown and Hoshino [4, Theorem 4.3] determined which connected cubic circulant graphs were well-covered. The last two authors and Watt refined this result to determine which of these graphs were Cohen-Macaulay (see [29, Theorem 5.2]), while Hoshino [17, Proposition 4.61] determined which ones were shellable. As a consequence of our computations, we observed that the Cohen-Macaulay connected cubic circulants were also vertex decomposable. More precisely, we have the following result, which summarizes the past theorems and our computations.

Theorem 7.5. *Let G be a connected cubic circulant graph. Then G is well-covered if and only if it is isomorphic to $C_4(1, 2)$, $C_6(1, 3)$, $C_6(2, 3)$, $C_8(1, 4)$, or $C_{10}(2, 5)$. In addition*

- (i) $C_4(1, 2)$ and $C_6(2, 3)$ are vertex decomposable.
- (ii) $C_6(1, 3)$, $C_8(1, 4)$, $C_{10}(2, 5)$ are Buchsbaum but not Cohen-Macaulay.

7.3. Well-covered circulants of small order. By a computer search, we have determined all well-covered circulant graphs on $3 \leq n \leq 16$ vertices. For each well-covered graph, we determined if it was vertex decomposable, shellable, Cohen-Macaulay, and/or Buchsbaum. We also determined if the circulant was connected or not, and whether or not it was 1-well-covered. Our computations were made using Sage [27] and *Macaulay2* [11]. The *Macaulay2* packages *EdgeIdeals* [10] and *SimplicialDecomposability* [5] were also used to carry out our experiments.

Table 1 contains the following information. Every circulant $G = C_n(a_1, \dots, a_t)$ in the table is well-covered. If there is a circulant $C_n(b_1, \dots, b_t) \cong C_n(a_1, \dots, a_t)$, we only list one circulant. A $*$ is used to indicate that the graph is disconnected. Because we have the implications,

$$\text{vertex decomposable} \Rightarrow \text{shellable} \Rightarrow \text{Cohen-Macaulay} \Rightarrow \text{Buchsbaum}$$

it is enough to know the strongest structure $C_n(a_1, \dots, a_t)$ possesses. We therefore write V if G is vertex decomposable, S if G is shellable but not vertex decomposable, B if G is Buchsbaum but not Cohen-Macaulay, and N if G has none of these properties. Finally, if G is 1-well-covered, we denote this in a separate column by a 1.

TABLE 1. Well-Covered circulant graphs up to order 16.

$C_3(1)$	V 1	$C_{11}(1, 2, 3, 4, 5)$	V 1	$C_{14}(2, 4, 6)^*$	V 1	$C_{16}(2, 4)^*$	V 1
$C_4(1)$	B	$C_{12}(3)^*$	N	$C_{14}(2, 4, 7)$	B	$C_{16}(2, 6)^*$	N
$C_4(2)^*$	V 1	$C_{12}(4)^*$	V 1	$C_{14}(1, 2, 3, 4)$	V 1	$C_{16}(2, 8)^*$	N
$C_4(1, 2)$	V 1	$C_{12}(6)^*$	V 1	$C_{14}(1, 2, 3, 7)$	B	$C_{16}(4, 8)^*$	V 1
$C_5(1)$	V 1	$C_{12}(1, 4)$	N	$C_{14}(1, 2, 4, 6)$	V 1	$C_{16}(1, 2, 4)$	B
$C_5(1, 2)$	V 1	$C_{12}(2, 4)^*$	N	$C_{14}(1, 2, 4, 7)$	B	$C_{16}(1, 2, 6)$	B 1
$C_6(2)^*$	V 1	$C_{12}(2, 6)^*$	N	$C_{14}(1, 2, 5, 6)$	N	$C_{16}(1, 4, 6)$	V 1
$C_6(3)^*$	V 1	$C_{12}(3, 4)$	B	$C_{14}(1, 2, 5, 7)$	B 1	$C_{16}(1, 4, 7)$	N
$C_6(1, 2)$	B	$C_{12}(3, 6)^*$	V 1	$C_{14}(1, 3, 5, 7)$	B	$C_{16}(1, 4, 8)$	S 1
$C_6(1, 3)$	B	$C_{12}(4, 6)^*$	V 1	$C_{14}(1, 4, 6, 7)$	B 1	$C_{16}(1, 6, 8)$	B
$C_6(2, 3)$	V 1	$C_{12}(1, 2, 6)$	B 1	$C_{14}(2, 4, 6, 7)$	V 1	$C_{16}(2, 4, 6)^*$	N
$C_6(1, 2, 3)$	V 1	$C_{12}(1, 3, 5)$	B	$C_{14}(1, 2, 3, 4, 5)$	V 1	$C_{16}(2, 4, 8)^*$	V 1
$C_7(1)$	B	$C_{12}(1, 3, 6)$	V 1	$C_{14}(1, 2, 3, 4, 6)$	V 1	$C_{16}(2, 6, 8)^*$	N 1
$C_7(1, 2)$	V 1	$C_{12}(1, 4, 6)$	B	$C_{14}(1, 2, 3, 4, 7)$	V 1	$C_{16}(1, 2, 3, 8)$	B 1
$C_7(1, 2, 3)$	V 1	$C_{12}(2, 3, 4)$	N	$C_{14}(1, 2, 3, 5, 7)$	B	$C_{16}(1, 2, 4, 7)$	B
$C_8(2)^*$	N	$C_{12}(2, 3, 6)$	B 1	$C_{14}(1, 2, 3, 6, 7)$	B	$C_{16}(1, 2, 5, 8)$	B
$C_8(4)^*$	V 1	$C_{12}(2, 4, 6)^*$	V 1	$C_{14}(1, 2, 4, 6, 7)$	V 1	$C_{16}(1, 2, 6, 7)$	N
$C_8(1, 2)$	V 1	$C_{12}(3, 4, 6)$	B 1	$C_{14}(1, 2, 5, 6, 7)$	V 1	$C_{16}(1, 2, 6, 8)$	N
$C_8(1, 3)$	B	$C_{12}(1, 2, 3, 4)$	V 1	$C_{14}(1, 2, 3, 4, 5, 6)$	B	$C_{16}(1, 2, 7, 8)$	B
$C_8(1, 4)$	B	$C_{12}(1, 2, 4, 5)$	B	$C_{14}(1, 2, 3, 4, 5, 7)$	B 1	$C_{16}(1, 3, 5, 7)$	B
$C_8(2, 4)^*$	V 1	$C_{12}(1, 2, 4, 6)$	V 1	$C_{14}(1, 2, 3, 4, 6, 7)$	V 1	$C_{16}(1, 4, 6, 8)$	B
$C_8(1, 2, 3)$	B	$C_{12}(1, 3, 4, 5)$	B 1	$C_{14}(1, 2, 3, 4, 5, 6, 7)$	V 1	$C_{16}(1, 4, 7, 8)$	B 1
$C_8(1, 2, 4)$	V 1	$C_{12}(1, 3, 4, 6)$	B	$C_{15}(3)^*$	V 1	$C_{16}(2, 4, 6, 8)^*$	V 1
$C_8(1, 3, 4)$	B 1	$C_{12}(1, 3, 5, 6)$	B 1	$C_{15}(5)^*$	V 1	$C_{16}(1, 2, 3, 4, 5)$	V 1
$C_8(1, 2, 3, 4)$	V 1	$C_{12}(1, 4, 5, 6)$	V 1	$C_{15}(1, 5)$	N	$C_{16}(1, 2, 3, 4, 6)$	V 1
$C_9(3)^*$	V 1	$C_{12}(2, 3, 4, 6)$	V 1	$C_{15}(3, 5)$	N	$C_{16}(1, 2, 3, 6, 8)$	B 1
$C_9(1, 3)$	B	$C_{12}(1, 2, 3, 4, 5)$	B	$C_{15}(3, 6)^*$	V 1	$C_{16}(1, 2, 3, 7, 8)$	B
$C_9(1, 2, 3)$	V 1	$C_{12}(1, 2, 3, 4, 6)$	V 1	$C_{15}(1, 2, 3)$	B	$C_{16}(1, 2, 4, 5, 8)$	B
$C_9(1, 2, 4)$	B	$C_{12}(1, 2, 3, 5, 6)$	B	$C_{15}(1, 3, 5)$	B	$C_{16}(1, 2, 4, 6, 7)$	N
$C_9(1, 2, 3, 4)$	V 1	$C_{12}(1, 2, 4, 5, 6)$	B 1	$C_{15}(1, 3, 6)$	V 1	$C_{16}(1, 2, 4, 6, 8)$	V 1
$C_{10}(2)^*$	V 1	$C_{12}(1, 3, 4, 5, 6)$	B 1	$C_{15}(1, 4, 6)$	N	$C_{16}(1, 2, 4, 7, 8)$	B 1
$C_{10}(5)^*$	V 1	$C_{12}(1, 2, 3, 4, 5, 6)$	V 1	$C_{15}(3, 5, 6)$	V 1	$C_{16}(1, 2, 6, 7, 8)$	V 1
$C_{10}(1, 4)$	N	$C_{13}(1, 3)$	B	$C_{15}(1, 2, 3, 6)$	N	$C_{16}(1, 3, 4, 5, 7)$	N
$C_{10}(2, 4)^*$	V 1	$C_{13}(1, 5)$	V 1	$C_{15}(1, 2, 3, 7)$	B 1	$C_{16}(1, 3, 5, 7, 8)$	B 1
$C_{10}(2, 5)$	B	$C_{13}(1, 2, 4)$	B	$C_{15}(1, 2, 5, 6)$	B	$C_{16}(1, 2, 3, 4, 5, 6)$	V 1
$C_{10}(1, 2, 3)$	V 1	$C_{13}(1, 2, 5)$	B	$C_{15}(1, 3, 4, 5)$	B 1	$C_{16}(1, 2, 3, 4, 5, 7)$	B 1
$C_{10}(1, 2, 4)$	V 1	$C_{13}(1, 3, 4)$	B 1	$C_{15}(1, 3, 4, 6)$	B 1	$C_{16}(1, 2, 3, 4, 5, 8)$	V 1
$C_{10}(1, 2, 5)$	B	$C_{13}(1, 2, 3, 4)$	V 1	$C_{15}(1, 3, 5, 6)$	B	$C_{16}(1, 2, 3, 4, 6, 8)$	V 1
$C_{10}(1, 3, 5)$	B	$C_{13}(1, 2, 3, 5)$	V 1	$C_{15}(1, 4, 5, 6)$	V 1	$C_{16}(1, 2, 3, 4, 7, 8)$	B
$C_{10}(1, 4, 5)$	V 1	$C_{13}(1, 2, 3, 6)$	B	$C_{15}(1, 2, 3, 4, 5)$	V 1	$C_{16}(1, 2, 3, 5, 6, 7)$	B
$C_{10}(2, 4, 5)$	V 1	$C_{13}(1, 2, 3, 4, 5)$	V 1	$C_{15}(1, 2, 3, 5, 6)$	B	$C_{16}(1, 2, 3, 5, 6, 8)$	V 1
$C_{10}(1, 2, 3, 4)$	B	$C_{13}(1, 2, 3, 4, 5, 6)$	V 1	$C_{15}(1, 2, 3, 5, 7)$	V 1	$C_{16}(1, 2, 3, 5, 7, 8)$	B
$C_{10}(1, 2, 3, 5)$	B 1	$C_{14}(2)^*$	N	$C_{15}(1, 2, 4, 5, 7)$	B	$C_{16}(1, 2, 4, 6, 7, 8)$	V 1
$C_{10}(1, 2, 4, 5)$	V 1	$C_{14}(7)^*$	V 1	$C_{15}(1, 3, 4, 5, 6)$	V 1	$C_{16}(1, 3, 4, 5, 7, 8)$	B 1
$C_{10}(1, 2, 3, 4, 5)$	V 1	$C_{14}(1, 6)$	N	$C_{15}(1, 2, 3, 4, 5, 6)$	V 1	$C_{16}(1, 2, 3, 4, 5, 6, 7)$	B
$C_{11}(1, 2)$	B	$C_{14}(2, 4)^*$	V 1	$C_{15}(1, 2, 3, 4, 5, 7)$	B 1	$C_{16}(1, 2, 3, 4, 5, 6, 8)$	V 1
$C_{11}(1, 3)$	B	$C_{14}(1, 2, 5)$	B	$C_{15}(1, 2, 3, 4, 6, 7)$	B	$C_{16}(1, 2, 3, 4, 5, 7, 8)$	B 1
$C_{11}(1, 2, 3)$	V 1	$C_{14}(1, 4, 6)$	B	$C_{15}(1, 2, 3, 4, 5, 6, 7)$	V 1	$C_{16}(1, 2, 3, 5, 6, 7, 8)$	B 1
$C_{11}(1, 2, 4)$	B	$C_{14}(1, 4, 7)$	B	$C_{16}(4)^*$	N	$C_{16}(1, 2, 3, 4, 5, 6, 7, 8)$	V 1
$C_{11}(1, 2, 3, 4)$	V 1	$C_{14}(1, 6, 7)$	B 1	$C_{16}(8)^*$	V 1		

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